# From Kantorovitch Problem to Linear Sum Assignment Problem

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**Abstract**

The goal of this technical report is to detail the link between the Linear Sum Assignment Problem (LSAP) and the Kantorovitch Problem (KP) also called Optimal Transport. This relation is not new and is reported in [5]

## 1 Introduction

The goal of this technical report is to draw a link[1] between the Linear Sum Assignment Problem (LSAP) [2] and the Kantorovitch Problem (KP) [5]. The link between both problems is strong because KP is a generalization of LSAP. LSAP is a special case of the KP.

Consequently, fast heuristics for the KP can be applied to LSAP.

## 2 Linear Sum Assignment Problem

$V$ is a set of vertices. A vertex $u \in V$ is an arbitrary object. $|V|$ is the size of the set $V$. LSAP is a matching problem between two sets of vertices $V_1$ and $V_2$ of equal size such that $|V_1| = |V_2| = N$. The objective of LSAP is to find correspondences between two sets $V_1$ and $V_2$. A solution of LSAP is defined as a subset of possible correspondences $\mathcal{V} \subseteq V_1 \times V_2$, represented by a binary assignment matrix $\mathbf{V} \in \{0,1\}^{|N \times N}$. If $u_i \in V_1$ matches $u_k \in V_2$, then $\mathbf{V}_{i,k} = 1$, and $\mathbf{V}_{i,k} = 0$ otherwise. We denote by $\mathbf{v} \in \{0,1\}^{N^2}$, a column-wise vectorized replica of $\mathbf{V}$. With this notation, matching problems can be expressed as finding the assignment vector $\hat{\mathbf{v}}$ that minimizes a cost function $C(V_1, V_2, \mathbf{v})$ as follows:

[1] `https://www.youtube.com/watch?v=Vyjf7TnUYKk`

**Problem 1.** Linear Sum Assignment Problem (LSAP)

$$\hat{\mathbf{v}} = \underset{\mathbf{v}}{\operatorname{argmin}} \quad C(V_1, V_2, \mathbf{v}) \tag{1a}$$

$$\text{subject to} \quad \mathbf{v} \in \{0,1\}^{N^2} \tag{1b}$$

$$\sum_{u_i \in V_1} \mathbf{v}_{i,k} = 1 \quad \forall u_k \in V_2 \tag{1c}$$

$$\sum_{u_k \in V_2} \mathbf{v}_{i,k} = 1 \quad \forall u_i \in V_1 \tag{1d}$$

where equations (1c),(1d) induces the matching constraints, thus making $\mathbf{v}$ an assignment vector. Therefore, $\mathbf{V}$ is a permutation matrix where rows and column sum to one and containing values in $\{0,1\}$.

The function $C(V_1, V_2, \mathbf{v})$ measures the dissimilarity of vertices, and is composed of a first order dissimilarity function $c(u_i, u_k) \in \mathbb{R}_+$ for a vertex pair $u_i \in V_1$ and $u_k \in V_2$. The objective function of LSAP is defined as:

$$C(V_1, V_2, \mathbf{v}) = \sum_{u_i \in V_1} \sum_{u_k \in V_2} c(u_i, u_k) \cdot \mathbf{v}_{i,k} \tag{2}$$

In essence, the cost accumulates all the dissimilarity values that are relevant to the assignment. $C$ is a linear function according to the variable $\mathbf{v}$. The problem has been proven to be polynomial with a worst case complexity $O(N^3)$ [2]. One famous exact method for the LSAP is the Hungarian method [2].

Dissimilarities $\{\ c(u_i, u_k) \mid u_i \in V_1 \text{ and } u_k \in V_2\ \}$ can be stored in a vector $\mathbf{cv} \in \mathbb{R}_+^{N^2}$. Therefore the objective function can be written as follows :

$$C(V_1, V_2, \mathbf{v}) = \mathbf{cv}^T \cdot \mathbf{v} \tag{3}$$

This problem (Problem 1) is a discrete problem with an *argmin* operator and it can be written as a permutation problem. $\phi$ is a permutation function defined as $\phi : \{1, 2, 3, \cdots, N\} \to \{1, 2, 3, \cdots, N\}$. $\Phi$ is a set of all permutations.

**Problem 2.** Permutation problem

$$\hat{\phi} = \underset{\phi \in \Phi}{\operatorname{argmin}} \quad \sum_{u_i \in V_1} \sum_{u_k \in V_2} c(u_i, u_{\phi(i)}) \cdot \mathbf{v}_{i,\phi(i)} \tag{4}$$

# 3 Kantorovitch Problem

$X = \{x_i\}_{i=1}^{|X|}$ and $Y = \{y_j\}_{j=1}^{|Y|}$ are two sets with $x_i$ and $y_j \in \mathbb{R}^d$. Associated with these two sets, we denote by $Pr(x_i) = a_i$ and $Pr(y_j) = b_j$. $Pr(x_i)$ is the probability to observe $x_i$. A solution of KP is matrix $\mathbf{P} \in \mathbb{R}_+^{|X| \times |Y|}$. We denote by $\mathbf{p} \in \mathbb{R}_+^{|X|.|Y|}$, a column-wise vectorized replica of $\mathbf{P}$. The KP is defined as follows :

**Problem 3.** Kantorovitch Problem (KP)

$$\hat{\mathbf{p}} = \underset{\mathbf{P}}{\text{argmin}} \sum_{x_i \in X} \sum_{y_j \in Y} c(x_i, y_j) \cdot \mathbf{p}_{i,j} \tag{5a}$$

$$\text{subject to} \quad \mathbf{p} \in \mathbb{R}_+^{|X|.|Y|} \tag{5b}$$

$$\sum_{a_i \in X} \mathbf{p}_{i,j} = b_j \quad \forall b_j \in Y \tag{5c}$$

$$\sum_{b_j \in Y} \mathbf{p}_{i,j} = a_i \quad \forall a_i \in X \tag{5d}$$

$$\sum_{a_i \in X} a_i = \sum_{b_j \in Y} b_j \tag{5e}$$

There exists a solution to KP only if $\sum_{a_i \in X} a_i = \sum_{b_j \in Y} b_j$.

# 4 From KP to LSAP

In this section, we want to show the link between KP and LSAP.

Let us define a special case of KP where:

1. $|X| = |Y| = N$
2. $a_i = b_j = 1 \quad \forall a_i \in X \quad \forall b_j \in Y$.

Now we define the Kantorovitch Problem in this special case.

**Problem 4.** Special Case Kantorovitch Problem (SCKP)

$$\hat{\mathbf{p}} = \underset{\mathbf{P}}{\text{argmin}} \sum_{a_i \in X} \sum_{b_j \in Y} c(x_i, y_j) \cdot \mathbf{p}_{i,j} \tag{6a}$$

$$\text{subject to} \quad \mathbf{p} \in \mathbb{R}_+^{|X|.|Y|} \tag{6b}$$

$$\sum_{a_i \in X} \mathbf{p}_{i,j} = 1 \quad \forall b_j \in Y \tag{6c}$$

$$\sum_{b_j \in Y} \mathbf{p}_{i,j} = 1 \quad \forall a_i \in X \tag{6d}$$

SCKP has a worst case complexity in $O(N^3)$ [5].

$\mathbf{P}$ is a doubly-stochastic matrix. $\mathbf{P}$ is positive and rows and columns sum to one. $\mathcal{P}$ is the set of all doubly-stochastic matrices. The set of all permutations $\Phi$ is smaller than $\mathcal{P}$ because it is the intersection between two sets :

$$\Phi = \mathcal{P} \cap \{0,1\}^{N \times N} \tag{7}$$

$$\Phi \subseteq \mathcal{P} \tag{8}$$

Permutation matrices are doubly-stochastic matrices. The inverse is not true. LSAP operates on a smaller set of variables than SCKP. $\mathbf{P} \in \mathbb{R}_+^{N \times N}$ is doubly-stochastic matrix where rows and columns sum to one consequently any value $\mathbf{P}_{i,j}$ must be smaller or equal to one. Therefore the domain of $\mathbf{P}$ can be refined to $\mathbf{P} \in [0,1]^{N \times N}$.

Here are the main ideas linking the SCKP to LSAP :

1. SCKP minimises a linear function on the set of doubly-stochastic matrices.
2. LSAP minimises a linear function on the set of permutation matrices.

3. The set of permutation matrices is a subset of the set of doubly-stochastic matrices.
4. The solution space of SCKP is a bounded polytope so it is a polyhedra.
5. The polyhedra is convex. Convex means when you link two points in the convex set you stay in the set.
6. Extreme points of a polyhedra are vertices.
7. Minimizing a linear function over a convex set implies that there exits an optimal solution of the problem that is an extreme point (a vertex) of the polyhedra.
8. Consequently, SCKP has at least one optimal solution that is an extreme point of the polyhedra.
9. The Birkhoff [4] theorem says that extreme points of the set of doubly-stochastic matrices belong to the set of permutation matrices.
10. A consequence of the Birkhoff [4], there is an optimal solution of SCKP that is an optimal solution of LSAP.
11. Issue : Many optimal solutions of SCKP can exist. All optimal solutions are not permutation matrices.

# 5 From relaxed LSAP to KP

Let us relax the LSAP to the continuous domain.

**Problem 5.** Relaxed Linear Sum Assignment Problem

$$\hat{\mathbf{v}} = \underset{\mathbf{v}}{\operatorname{argmin}} \sum_{u_i \in V_1} \sum_{u_k \in V_2} c(u_i, u_k) \cdot \mathbf{v}_{i,k} \tag{9a}$$

$$\text{subject to} \quad \mathbf{v} \in [0,1]^{N^2} \tag{9b}$$

$$\sum_{u_i \in V_1} \mathbf{v}_{i,k} = 1 \quad \forall u_k \in V_2 \tag{9c}$$

$$\sum_{u_k \in V_2} \mathbf{v}_{i,k} = 1 \quad \forall u_i \in V_1 \tag{9d}$$

The Relaxed LSAP is a special case of the Kantorovich problem named SCKP in our paper. Fortunately, the optimal solution of the continuous relaxation of the LSAP gives the optimal solution the LSAP. It can be seen as a consequence of the Birkhoff theorem that ensures doubly-stochastic matrices are the convex envelope of permutation matrices.

The Sinkhorn algorithm [6, 1] is fast heuristic for the KP. Therefore, LSAP can be solved efficiently thanks to the Sinkhorn algorithm. Issue : The solution provided by the Sinkhorn algorithm is not mandatory a permutation matrix.

# 6 Conclusion

In the technical report, we show how the Linear Sum Assignment Problem can be seen as a special case of the Kantorovitch Problem. So, LSAP can be solved efficiently thanks to the Sinkhorn algorithm. Even though the original algorithm assumes only square matrices, the process can be generalized as shown in [3]

# References

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